JOURNAL OF APPROXIMATION THEORY 44, 230–240 (1985)

Uniqueness of Best Chebyshev Approximations in Spline Subspaces

HANS STRAUSS

Institut für Angewandte Mathematik, Universität Erlangen-Nürnberg, 8520 Erlangen, West Germany

Communicated by G. Meinardus

Received December 12, 1983; revised April 10, 1984

This paper deals with the problem of uniqueness of best Chebyshev approximations by subspaces of spline functions on compact subsets T of $\mathbb{R}$. Necessary and sufficient conditions ensuring uniqueness of best approximations are given and a characterization of strongly unique best approximations using best approximations on finite subsets of T is established. Moreover, problems where a best approximation is unique on an interval I but is not a unique best approximation on any finite subset are considered.

INTRODUCTION

Let T be a (nonempty) compact subset of $\mathbb{R}$ and $C(T)$ the space of real-valued functions on T, where $C(T)$ is normed by $\|f\| := \sup\{|f(x)|: x \in T\}$. Suppose that G is an n-dimensional subspace of $C(T)$ then the set of *best Chebyshev approximations* to a function f in $C(T)$ out of G is defined by

$$\{g_0 \in G: \|f - g_0\| = \inf\{\|f - g\|: g \in G\}\}.$$

This paper deals with an approximation problem where G are subspaces of spline functions $S_m(\Delta)$ of degree $m-1$ with k fixed knots, $n = m + k$. We study the relationship between best approximations for problems defined on compact sets T and finite subsets thereof. In particular, we consider problems where the best approximation is unique.

First, we study approximation problems on finite subsets T. Rice [10] has defined a "strict approximation" $s(f, T)$ which is a particular unique best Chebyshev approximation. If G is a spline subspace these strict approximations have been characterized in [15, 16]. Here we construct a subset $R \subset T$ such that R has at most $2n$ points and $s(f, T)$ is the (unique) strict approximation to f on R. Similar results are not true for

0021-9045/85 $3.00

approximation problems defined on an interval I, in general. Let f be a function on I which has a strongly unique best approximation s_0; then we can construct a subset R as above such that R contains at most $2n$ points and s_0 is a (strongly) unique best approximation to f on R. Strongly unique best approximations can be characterized by such subsets (see also Brosowski [2]). To determine a best approximation we compute strict approximations $s(f, T_i)$ on certain finite subsets T_i. If f has a strongly unique best approximation s_0, then we can define a sequence of subsets T_i such that $\{s(f, T_i)\}$ converges to s_0. Moreover, we give conditions where $\{s(f, T_i)\}$ converges if f has not a strongly unique best approximation. The above-mentioned finite subsets R which contain at most $2n$ points play an important role in this construction.

Finally we consider finite subsets which "fill out" an interval I and give examples where a best approximation is unique on I but is not unique on any finite subset. We shall show that this is true if f has not a strongly unique best approximation.

1. Preliminaries

Let a function f in $C(T)$ be given. We use the following notations: We denote by $E(f)$ the set of *extreme points* of the function f on T,

$$E(f) = \{x \in T: |f(x)| = \|f\|\}.$$

A function f is said to *alternate* on the points $t_1 < \cdots < t_h$ in T if $f(t_i)\, f(t_{i+1}) < 0$, $i = 1,..., h-1$ and we call points $t_1 < \cdots < t_h$ in T *alternating extreme points* of f if $\mu(-1)^i f(t_i) = \|f\|$, $i = 1,..., h$, $\mu \in \{-1, 1\}$.

If the subset U of T contains at least two alternating extreme points, then we count the number of alternations of f in U by

$$A_U(f) = \max\{p: \text{there exist } p+1 \text{ alternating extreme points of } f \text{ in the subset } U\}.$$

If U does not contain two alternating extreme points, we write $A_U(f) = 0$.

We also consider approximation problems defined on compact subsets $U \subset T$. Let $\|f\|_U := \sup\{|f(x)|: x \in U\}$. Then the set of best Chebyshev approximations to f out of G on U is defined by $\{g_0 \in G: \|f - g_0\|_U = \inf\{\|f - g\|_U: g \in G\}\}$, where G is an n-dimensional subspace of $C(T)$.

A subspace G satisfies the *Haar condition* if $g \in G$, $g(x) = 0$ at n distinct points of T implies $g \equiv 0$. In this case the best approximation is always unique. Moreover, there exists a subset U of T for the best approximation g_0 which contains $n+1$ points such that g_0 is also a unique best approximation from G to f on U.

If G does not satisfy the Haar condition we do not have similar results. In this paper we shall study problems concerning uniqueness of best approximations. Moreover, we consider the relationship between best approximations for problems defined on a set T and the best approximations on subsets of T.

We shall need the following notations: A subset $\{g_1,..., g_n\}$ of linearly independent functions of $C(T)$ is called a *weak Chebyshev system* if every function g in $G=\text{span}\{g_1,..., g_n\}$ has at mst $n-1$ sign changes on T. The subspace G is called a *weak Chebyshev subspace*. The subset $\{g_i\}_{i=1}^n$ is called a *complete weak Chebyshev system* if the subsets $\{g_i\}_{i=1}^k$ are weak Chebyshev systems for $k=1,..., n$. The subspace G is called a *complete weak Chebyshev subspace* if G contains a basis $\{g_i\}_{i=1}^n$ which is a complete weak Chebyshev system.

LEMMA 1.1. *Let G be an n-dimensional weak Chebyshev subspace of $C(T)$, where T is a compact subset of* $\mathbb{R}$.

(a) *Then G is a complete weak Chebyshev subspace.*

(b) *Given $-\infty=t_0<t_1<\cdots<t_{n-1}<t_n=\infty$ with $\{t_i\}_{i=1}^{n-1}\subset T$. Then there exists a nontrivial g in G such that*

$$(-1)^{i+1}g(x)\geqslant 0, \qquad x\in[t_{i-1}, t_i)\cap T, \qquad i=1,..., n.$$

For a proof of (a) and (b) see [13] and [4].

Suppose that g_0 is a best approximation from G to f on T. A subset S of the extreme points of $f-g_0$ is said to be a *critical point set* if g_0 is a best approximation to f on S but is not a best approximation to f on any proper subset of S (see [10]). A critical point set contains at most $n+1$ points. If G satisfies the Haar condition, then a critical point set has exactly $n+1$ points.

Now we shall consider subspaces of spline functions.

Let $\Delta=\{x_i\}_{i=1}^k$ with $a=x_0<x_1<\cdots<x_k<x_{k+1}=b$ be a partition of $I=[x_0, x_{k+1}]$. The subspace $S_m(\Delta)$ of *polynomial spline functions* of degree $m-1$ $(m\geqslant 2)$ with simple fixed knots at Δ is defined by

$$S_m(\Delta)=\{s\in C^{m-2}[a, b]: s|_{[x_i,x_{i+1}]}\in\Pi_{m-1}, i=0,..., k\},$$

where Π_{m-1} denotes all polynomials of degree $\leqslant m-1$. Moreover, we define the following subspaces:

Let I be an interval satisfying $(x_0, x_{k+1}) \subset I \subset [x_0, x_{k+1}]$ and let T be a compact subset of I. Then

$$S_m(I) = \{s : s \in S_m(\Delta):\\
\text{if } I = (x_0, x_{k+1}], \text{ then } s^{(i)}(x_0) = 0,\ i = 0,..., m-2,\\
\text{if } I = [x_0, x_{k+1}), \text{ then } s^{(i)}(x_{k+1}) = 0,\ i = 0,..., m-2,\\
\text{if } I = (x_0, x_{k+1}), \text{ then } s^{(i)}(x_0) = s^{(i)}(x_{k+1}) = 0,\\
i = 0,..., m-2\},$$

$$S_m(I, T) = \{s|_T : s \in S_m(I)\}.$$

A local basis of $S_m(I)$ will be very useful. Let the partition Δ be given. A partition $\tilde{\Delta} = \{x_i\}_{i=-m+1}^{n}$, $n = m+k$, with $x_{-m+1} < \cdots < x_0 < \cdots < x_{k+1} < \cdots < x_n$ is called an *extended partition associated with* Δ.

Suppose that M_i, $i = -m+1,..., k$ is the mth *order B-spline* associated with the knots $x_i,..., x_{i+m}$ (see [12, p. 118]). We also denote $M_i|_J$ by M_i, where J is a subset of $[x_{-m+1}, x_{m+k}]$. Then $S_m(I) = \text{span}\{M_{-m+1},..., M_k\}$ if $I = [x_0, x_{k+1}]$, $S_m(I) = \text{span}\{M_0,..., M_k\}$ if $I = (x_0, x_{k+1}]$ and $S_m(I) = \text{span}\{M_0,..., M_{k-m+1}\}$ if $I = (x_0, x_{k+1})$.

PROBLEM I. Let the partition $\tilde{\Delta} = \{x_i\}_{i=-m+1}^{n}$, $n \geq 1$, be given and let $\tilde{I} = (x_{-m+1}, x_n)$. Suppose that T is a compact subset of $[x_{-m+1}, x_n]$ such that $\dim S_m(\tilde{I}, T) = n$. Let f be a function in $C(T)$. Determine the best approximations from $S_m(\tilde{I}, T)$ to f on T.

Now we shall consider characterization theorems for best approximations to f (see [15]).

THEOREM 1.2. *Let Problem* I *be given, where* $T \subset (x_{-m+1}, x_n)$.

(a) *Then* s_0 *is a solution of Problem* I *if and only if there exists a subinterval* J_R *and a subset* $R = \{u_i\}_{i=p}^{q+1} \subset T \cap J_R$ *such that* $f - s_0$ *has alternating extreme points on* R, *where* R *and* J_R *satisfy*

$$J_R = \begin{cases} (x_{-m+1}, x_n) & \text{if } p = 1, q = n, \\ [x_{p-1}, x_n) & \text{if } p > 1, q = n, \\ (x_{-m+1}, x_{-m+q+1}] & \text{if } p = 1, q < n, \\ [x_{p-1}, x_{-m+q+1}] & \text{if } p > 1, q < n, q - p \geq m - 1 \end{cases} \tag{1.1}$$

and

$$u_i \in (x_{-m+i}, x_{i-1}), \qquad i = p+1,..., q.$$

(b) *The best approximations are uniquely determined on* J_R.

(c) *A subset $S \subset T$ is a critical point set of $f - s_0$ if and only if there exists a subinterval J_S corresponding to S such that S and J_S satisfy the properties of* (1.1).

If R is a critical point set of $f - s_0$, then we denote the subinterval J_R to be associated with R.

Remark. The subspace $\tilde{S}_m(I)$ is spanned by B-splines. But it is possible to derive characterization theorems for all kinds of boundary conditions from Theorem 1.1 (for details, see [15]). For example, if we set $n = m + k$ and $T = [x_0, x_{k+1}]$ we obtain the problem which was considered in [10] and [11] that is an approximation problem defined on $[x_0, x_{k+1}]$. The characterization of best approximations in these papers follows immediately from this theorem.

2. Uniqueness

First, we want to study the problem of uniqueness of best approximations in this section. Alternation properties of the error function are very important. Let s_0 be a solution of Problem I. If there exists a subinterval $[x_i, x_{i+j+m-1}]$ satisfying

$$A_{T \cap [x_i, x_{i+j+m-1}]}(f - s_0) < j, \tag{2.1}$$

then s_0 is nonunique. This result for the subspace $S_m(\Delta)$ has been proved in [14]. Using Lemma 1.1 it can be easily seen that the result is also true for Problem I.

Now, we shall need the following notation: A function g_0 is called a strongly unique best approximation from a subspace G of $C(I)$ to f if there exists a constant $k > 0$ such that

$$\|f - g\| \geqslant \|f - g_0\| + k \, \|g - g_0\| \qquad \text{for all} \quad g \text{ in } G.$$

Let Problem I be given where $T \subset \tilde{I}$. Suppose that s_0 is a solution. Then s_0 is a strongly unique best approximation to f on T if and only if

$$A_{T \cap (x_i, x_{i+j+m-1})}(f - s_0) \geqslant j \qquad \text{for all} \quad (x_i, x_{i+j+m-1}) \subset \tilde{I}. \tag{2.2}$$

It has been shown in [6] that these assertions are true if $G = S_m(\Delta)$. The result is deduced from a general characterization theorem for strongly unique best approximations in weak Chebyshev subspaces. The above result also follows from this theorem.

Now, we shall study strongly unique best approximations. We consider the relationship between best approximations for problems defined on sets and the best approximations on certain subsets thereof.

First, we shall give an inductive definition of a function. This construction plays an important role in the theory of strict approximations (see [15, 16]). For a definition of strict approximations see [10].

DEFINITION 2.1. Let Problem I be given. Suppose that T is a finite subset of $\tilde{I}_j$ where $\dim S_m(\tilde{I}, T) = n$. Set $G_0 = S_m(\tilde{I})$, $\tilde{I}_0 = \phi$ and $Z_0 = \{-m+1,..., n-m\}$. Then we define for $j \geq 1$ the following sequence of functions s_j: Let G_j be the set of best approximations to the function

$$f - (s_1 + \cdots + s_{j-1}) \qquad \text{(i.e. } f \text{ if } j = 1\text{) on } T_j = T \cap \{\tilde{I} \backslash \tilde{I}_{j-1}\}$$

out of span $\{\{M_i\}_{i \in Z_{j-1}}\}$ and let s_j be a function in G_j. Suppose that I_j is a subinterval of $\tilde{I} \backslash \tilde{I}_{j-1}$ which is associated with a critical point set R_j of $(f - (s_1 + \cdots + s_j))$. Let $\gamma_j := \|f - (s_1 + \cdots + s_j)\|_{T_j}$. Then we define $\tilde{I}_j = \tilde{I}_{j-1} \cup I_j$ and $Z_j = \{i \in Z_{j-1} : \{x : M_i(x) \neq 0\} \cap \tilde{I}_j = \phi\}$. This construction is continued until $Z_t = \phi$ for some t.

The function $s(f, T) = s_1 + \cdots + s_t$ is called the strict approximation of f on T. $\{(I_i, R_i, \gamma_i)\}_{i=1}^t$ is said to be *associated* with $s(f, T)$.

LEMMA 2.2. *Let $s(f, T)$ be defined as above. Then the following assertions will hold*:

(a) *$R = \bigcup_{i=1}^t R_i$ contains at most $2n$ points.*

(b) *Let $s|_R \equiv 0$ for some $s \in S_m(\tilde{I})$; then $s \equiv 0$.*

Proof. It can be easily seen that the assertions will hold for $n = 1$. We now proceed by induction on n. Suppose that the theorem is correct for $n-1$ and suppose that $S_m(\tilde{I})$ is a subspace defined on $\tilde{I} = (x_{-m+1}, x_n)$. It is obvious that 0 is a best approximation to $e = f - s(f, T)$ from $S_m(\tilde{I})$ on T. There exists a critical point set R_1 associated with a subinterval I_1. We only consider the case where $n = m + k$ and $I_1 = [x_{p-1}, x_{-m+q+1}] \subset [x_1, x_k]$, $q - p \geq m - 1$. Then we conclude from Theorem 1.2 that $R_1 = \{t_i\}_{i=p}^{q+1}$. It follows from properties which are proved in [15] that $S_m(\tilde{I}, R_1)$ satisfies the Haar condition and that $s|_{R_1} \equiv 0$ implies $s|_{I_1} \equiv 0$ for all $s \in S_m(I_1)$. Now we apply the induction hypothesis to the function $e = f - s(f, T)$ on the subintervals $\tilde{I}_2 = (x_{-m+1}, x_{p-1})$ and $\tilde{I}_3 = (x_{-m+q+1}, x_n)$. Using the construction of $s(f, T)$ it follows that the subsets $\tilde{R}_2 = R \cap \tilde{I}_2$ and $\tilde{R}_3 = R \cap \tilde{I}_3$ satisfy the properties of the lemma relative to $S_m(\tilde{I}_2)$ and $S_m(\tilde{I}_3)$, respectively. $\tilde{R}_2$ contains at most $2(p-1)$ points and $\tilde{R}_3$ at most $2(n-q)$ points. Hence R contains at most $2n - q + p$ points. Moreover, we obtain that $s|_{\tilde{R}_i} \equiv 0$ for $s \in S_m(\tilde{I}_i)$ implies $s|_{\tilde{I}_i} \equiv 0$ for all $s \in S_m(\tilde{I}_i)$, $i = 2, 3$. Therefore $s|_R \equiv 0$ implies $s|_{\tilde{I}_i} \equiv 0$ for all $s \in S_m(\tilde{I})$. The other cases of I_1 can be similarly shown.

The process of Definition 2.1 cannot be carried out on a compact set T, in general. For example, let $T=[x_{-m+1}, x_n]$ in Problem I; then it is possible that $\{x_{-m+1}\}$ or $\{x_n\}$ is a critical point set. Then the best approximation is uniquely determined on one point, in general. Therefore we consider a modified subset. Let Problem I be given and $T \subset (x_{-m+1}, x_n)$. Then

$$\tilde{I}_\varepsilon = (x_{-m+1}, x_n) \setminus \bigcup_{i=-m+2}^{n-1} \{(x_i - \varepsilon, x_i) \cup (x_i, x_i + \varepsilon)\}$$

for some $\varepsilon > 0$ and $\tilde{T}_\varepsilon = T \cap \tilde{I}_\varepsilon$.

Using Theorem 1.2 we can show that the construction of Definition 2.1 is also possible for the subset $\tilde{T}_\varepsilon$, $\varepsilon > 0$ and sufficiently small. Hence we obtain a function $s(f, \tilde{T}_\varepsilon) = \sum_{i=1}^{t} s_i$ such that s_j is a best approximation to $f - (s_1 + \cdots + s_{j-1})$ on $\tilde{T}_\varepsilon \cap \{\tilde{I} \setminus \tilde{I}_{j-1}\}$ for all $j = 1, \ldots, t$, where the partition $\{(I_i, R_i, \gamma_i)\}_{i=1}^{t}$ corresponds to $\tilde{T}_\varepsilon$. The functions $s(f, \tilde{T}_\varepsilon)$ can be considered as strict approximations on $\tilde{T}_\varepsilon$ (See [16]).

We shall use this construction to characterize strongly unique best approximations.

THEOREM 2.3. *Let Problem* I *be given and* $T \subset \tilde{I}$. *Suppose that* s_0 *is a strongly unique best approximation to* f. *Then the following assertions will hold*:

(a) *There exists a* $d > 0$ *such that* $s_0 = s(f, \tilde{T}_\varepsilon)$ *for all* $0 < \varepsilon \leq d$. *Corresponding to all functions* $s(f, \tilde{T}_\varepsilon)$, $0 < \varepsilon \leq d$, *we have a partition* $\{(I_i, R_i, \gamma_i)\}_{i=1}^{t}$, *where* $\gamma_i = \gamma_{i+1}$, $i = 1, \ldots, t-1$.

(b) *The function* s_0 *is a unique best approximation to* f *on* $R = \bigcup_{i=1}^{t} R_i$.

Proof. It follows from (2.2) that exists a constant $d > 0$ such that

$$A_{\tilde{T}_\varepsilon \cap (x_i, x_{i+j+m-1})}(f - s_0) \geq j \qquad \text{for all } (x_i, x_{i+j+m-1}) \subset \tilde{I}, \tag{2.3}$$

where $0 < \varepsilon \leq d$. We conclude from Theorem 1.2 that 0 is a best approximation from $S_m(\tilde{I})$ to $f - s_0$ on $\tilde{T}_\varepsilon$. Moreover, there exists a critical point set which is associated with a subinterval I_1. We only consider the case where $I_1 = [x_{p-1}, x_{-m+1+q}]$. Then it follows from (2.3) that

$$A_{\tilde{T}_\varepsilon \cap (x_{-m+1}, x_{p-1})}(f - s_0) \geq p - 1$$

and

$$A_{\tilde{T}_\varepsilon \cap (x_{-m+1+q}, x_n)}(f - s_0) \geq n - q.$$

Hence there exists a critical point set of $f-s_0$ associated with a subinterval I_2 in $\tilde{I}\backslash I_1$ relative to $S_m(\tilde{I}\backslash I_1)$. Moreover, we see that $\gamma_1=\gamma_2$, where $\gamma_i=\|f-s_0\|_{\tilde{T}_\varepsilon\cap I_i}$, $i=1, 2$. Using these arguments we are able to apply the construction of Definition 2.1 to the function $f-s_0$ on $\tilde{T}_\varepsilon$ and we obtain a partition $\{(I_i, R_i, \gamma_i)\}_{i=1}^{t}$, where $\gamma_i=\gamma_{i+1}$, $i=1,..., t-1$. It follows from these properties that $s_0=s(f, \tilde{T}_\varepsilon)$ for all $0<\varepsilon\leqslant d$. Moreover, we conclude from the construction that s_0 is a unique best approximation to f on R.

DEFINITION 2.4. Let $f\in C(T)$ and $s_0\in S_m(\tilde{I}, T)$. If there exists a partition $\{(I_i, R_i)\}_{i=1}^{t}$ satisfying the conditions of Theorem 2.3(a), then we say $\{(I_i, R_i)\}_{i=1}^{t}$ is to be associated with s_0.

It follows from Lemma 2.2 that $R=\bigcup_{i=1}^{t} R_i$ contains at most $2n$ points and $s|_R\equiv 0$ implies $s\equiv 0$ for all $s\in S_m(\tilde{I})$, where $|(f-s_0)(t)|=\|f-s_0\|$ for all $t\in R$.

THEOREM 2.5. *Let Problem* I *be given and* $T\subset\tilde{I}$. *Then the following assertions are equivalent*:

(a) *The function f in $C(T)$ has a strongly unique best approximation* s_0.

(b) *There exists a partition $\{(I_i, R_i)\}_{i=1}^{t}$ which is associated with* s_0.

Proof. If f has a strongly unique best approximation s_0, then the assertions of (b) follow from Theorem 2.3. Assume to the contrary that (b) is satisfied. Then it follows from Theorem 1.2 that s_0 is a unique best approximation to f on $R=\bigcup_{i=1}^{t} R_i$. Hence we conclude from [1] and the properties of R that s_0 is strongly unique.

Remark. It has been shown in [2] that strongly unique best approximations can be characterized by finite subsets satisfying properties as above.

These results are important to the computation of best Chebyshev approximations. In [16, 17] the following algorithm is studied:

ALGORITHM 2.6. *Let Problem* I *be given and* $T\subset\tilde{I}$. *Let f be a function of $C(T)$. Suppose that $\tilde{T}_\varepsilon=T\cap\tilde{I}_\varepsilon$ is a compact subset of $\tilde{I}$ satisfying* $\dim S_m(\tilde{I}, \tilde{T}_\varepsilon)=n$ *and T_0 is a finite subset of $\tilde{T}_\varepsilon$ such that* $\dim S_m(\tilde{I}, T_0)=n$. *At the* ith *step is defined a finite subset T_i of $\tilde{T}_\varepsilon$ and $s(f, T_i)$ is a best approximation from $S_m(I)$ on T_i as defined in Definition* 2.1. *Let $\{I_{ij}\}_{j=1}^{t_i}$ be a partition of $\tilde{I}$ corresponding to $s(f, T_i)$. Suppose that y_{ij} is a point of $\tilde{T}_\varepsilon\cap I_{ij}$ such that*

$$|(f-s(f, T_i))(y_{ij})|\geqslant|(f-s(f, T_i))(x)| \qquad \textit{for all} \quad x\in\tilde{T}_\varepsilon\cap I_{ij},\ j=1,..., t_i.$$

Then T_{i+1} is given by $T_i\cup\{y_{ij}\}_{j=1}^{t_i}$.

The algorithm defines a sequence of finite subsets T_i and determines best approximations $s(f, T_i)$ on T_i. Corresponding to $s(f, T_i)$ we have partitions $\{(I_{ij}, R_{ij}, \gamma_{ij})\}_{j=1}^{t_i}$. The sets $R^i = \bigcup_{j=1}^{t_i} R_{ij}$ play a similar role as the so-called "references" in the Remez algorithm for subspaces satisfying the Haar condition. It follows from Lemma 2.2 that R^i contains at most $2n$ points.

THEOREM 2.7. *Let Problem* I *be given and* $T \subset \tilde{I}$. *Suppose that* f *has a strongly unique best approximation* s_0. *Then there exists a* $d > 0$ *such that the following assertions will hold*:

(a) *The sequence* $s(f, T_i)$ *of Algorithm* I *converges to* s_0 *on* T *for all* $0 < \varepsilon \leqslant d$.

(b) *Let* $\{(I_{ij}, R_{ij}, \gamma_{ij})\}_{j=1}^{t_i}$ *be partitions corresponding to* $s(f, T_i)$ *and let* $R^i = \bigcup_{j=1}^{t_i} R_{ij}$. *Then each cluster point* R *of* $\{R^i\}$ *satisfies the following property*: $s|_R \equiv 0$ *implies* $s \equiv 0$ *for all* $s \in S_m(\tilde{I})$.

(c) *The function* s_0 *is a unique best approximation from* $S_m(\tilde{I}, R)$ *to* f *on* R.

Proof. It follows from arguments as in [16] that $s(f, T_i)$ converges on each set $\tilde{T}_\varepsilon$ to a best approximation. Now we conclude from Theorem 2.3 that there is a $d > 0$ such that $s_0 = s(f, \tilde{T}_\varepsilon)$ for all $0 < \varepsilon \leqslant d$. Hence the sequence $s(f, T_i)$ converges to s_0 for all $0 < \varepsilon \leqslant d$. The other assertions follow as in [16].

Remark. If the cluster points of $\{R_{ij}\}$ do not satisfy the properties of Theorem 2.7(b) then there arises problems in Algorithm 2.6. Then it is only possible to determine the best approximation on $\tilde{T}_\varepsilon$, in general.

If f has not a unique best approximation then we obtain the following theorem.

THEOREM 2.8. *Let Problem* I *be given and* $T \subset \tilde{I}$. *Suppose that there is a function* $s_0 \in S_m(\tilde{I})$ *and* $d > 0$ *such that* $s_0 = s(f, \tilde{T}_\varepsilon)$ *for all* $0 < \varepsilon \leqslant d$. *Then the assertions of Theorem* 2.7(a), (b) *are true. Let* R *be a cluster point as in Theorem* 2.7 *then* s_0 *is a strict approximation from* $S_m(\tilde{I}, R)$ *to* f *on* R.

This theorem can be proved as Theorem 2.7. The function s_0 can be considered as a strict approximation for an approximation problem defined on an interval.

3. NONUNIQUENESS ON FINITE SUBSETS

In this section we shall study approximation problems on an interval and on subsets which "fill out" the interval. We shall need the following notation: Let X be a compact subset of $\mathbb{R}$ and Y a subset of X. We define the *density* of Y in X by $d(X, Y) = \max_{x \in X} \inf_{y \in Y} |(x - y)|$.

THEOREM 3.1. *Let the interval $I = [a, b]$ be given, let G be an n-dimensional subspace of $C(I)$ and f be in $C(I)$. Suppose that s_0 is a unique best approximation from G to f which is not strongly unique. Then there exist finite subsets R_ε of I for all $\varepsilon > 0$ satisfying $d(I, R_\varepsilon) < \varepsilon$ such that s_0 is a best approximation from G to f on I but s_0 is not unique on R_ε.*

Proof. Let R_1 be a critical point set of $f - s_0$. Then s_0 is a best approximation from G to f on R_1. Let R_ε in I be a finite subset such that $d(I, R_\varepsilon) < \varepsilon$, $R_1 \subset R_\varepsilon$ and $s|_{R_\varepsilon} \equiv 0$ implies $s \equiv 0$ for all $s \in S_m(I)$. Then it follows that s_0 is a best approximation to f on R_ε which is not uniquely determined. Assume to the contrary that s_0 is a unique best approximation on R_ε. Then we conclude from [1] that s_0 is a strongly unique best approximation on I. This contradiction proves the theorem.

Remark 3.2. There exist unique best approximations which are not strongly unique. We shall give a simple example and consider Problem I.

Let the partition $\Delta = \{x_1\}$ be given where $x_0 = 0$, $x_1 = 1$, $x_2 = 2$. Suppose that $S_2(I)$, $I = [0, 2]$, corresponds to Δ and the function f is defined as follows: $f(x) = 8(x^2 - x) + 1$ for $x \in [0, 1]$ and $f(x) = -2x^2 + 4x - 1$ for $x \in (1, 2]$. It is obvious that 0 is a unique best approximation to f. But 0 is not strongly unique since $A_{(x_1, x_3)}(f) = 0$.

A characterization of uniqueness for problems defined on $[a, b]$ has been given in [9] and for subspaces of generalized Tchebycheffian splines in [8].

Remark 3.3. Dunham [5] has established a sufficient condition such that the best approximations on all sufficiently dense subsets must be unique. Moreover, a special approximation problem is given which satisfies properties as in Theorem 3.1.

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